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THE CORRELATION STRUCTURE OF RANDOMLY ORIENTED 1,2,......N DIMENSIONAL WAVES

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13. ABSTRACT *(Maximum 200 words)*

Here, a wave is defined as a periodic function in one or more dimensions. The superimposition of waves can be used to generate a Gaussian field which in turn can be used to simulate a meteorological field. For this field to resemble a natural field, the correlation structure in the Gaussian field must resemble the natural field correlation structure.

For N dimensional waves of a single wavelength that are uniformly distributed with respect to phase and orientation, the homogeneous isotropic correlation can be found by integrating the one-dimensional correlation weighted by the density of the N dimensional direction cosine distribution. This has been done for the sine/cosine, sawtooth, and triangular waves and graphs and equations are given. The triangular wave correlation closely resembles the sine/cosine correlation with possible implications for objective analysis.

When coordinate systems are stretched and waves of different wavelengths and types are combined, a wide variety of nonhomogenous anisotropic correlation structure results.

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1. MOTIVATION

A family of simulation models has been developed to provide Monte Carlo simulation of atmospheric variables in time and space. These models are based on the superimposition of waves to approximate a standardized Gaussian field. Three parts of these models must function accurately if the generated fields are to imitate the atmosphere:

a. Convergence to a normal distribution of the superimposed waves.

b. Transformation of the normal distribution to the variables' climatological distribution.

c. Correlation structure in the generated Gaussian field must closely approximate the variables' equivalent normal correlation structure in time and space.

It is the third requirement that this report addresses. Furthermore, the results are applicable to other statistical atmospheric problems including objective analysis, inferential testing, coverage distributions, and extremes.

2. WAVE GENERATORS

Here, a wave is defined as a periodic function in one or more dimensions:

$$\text{Wave}(u) = \text{Wave}(u + L) \qquad (1)$$

where wave is some arbitrary waveform, e.g., a cosine or a sawtooth, L is the wavelength and (1) is true for any value of u. Initial consideration is limited to plane waves, that is, u is a linear function.

In 3 dimensions; $u = c_x x + c_y y + c_z z + u_0$ (2)

In D dimensions: $u = \underline{c} \cdot \underline{x} + u_0$ (3)

where the c's are direction cosines of a line (the cosine of the angle between each axis and the line) with positive/negative orientation through the origin, $\underline{c}\ \underline{x}$ is a dot product between $\underline{c}$, a row vector of direction cosines, and $\underline{x}$, a column vector, and u_0, $0 \le u_0 < L$, gives the phase at the origin. Equation (3) defines a plane perpendicular to the line specified by the direction cosines. As illustrated in Figure 1, the value of the wave is the same everywhere on this plane.

Equation (1) along with (2) or (3) define the value, v, of the wave everywhere in the space considered. If a set of such waves, each with different direction cosines

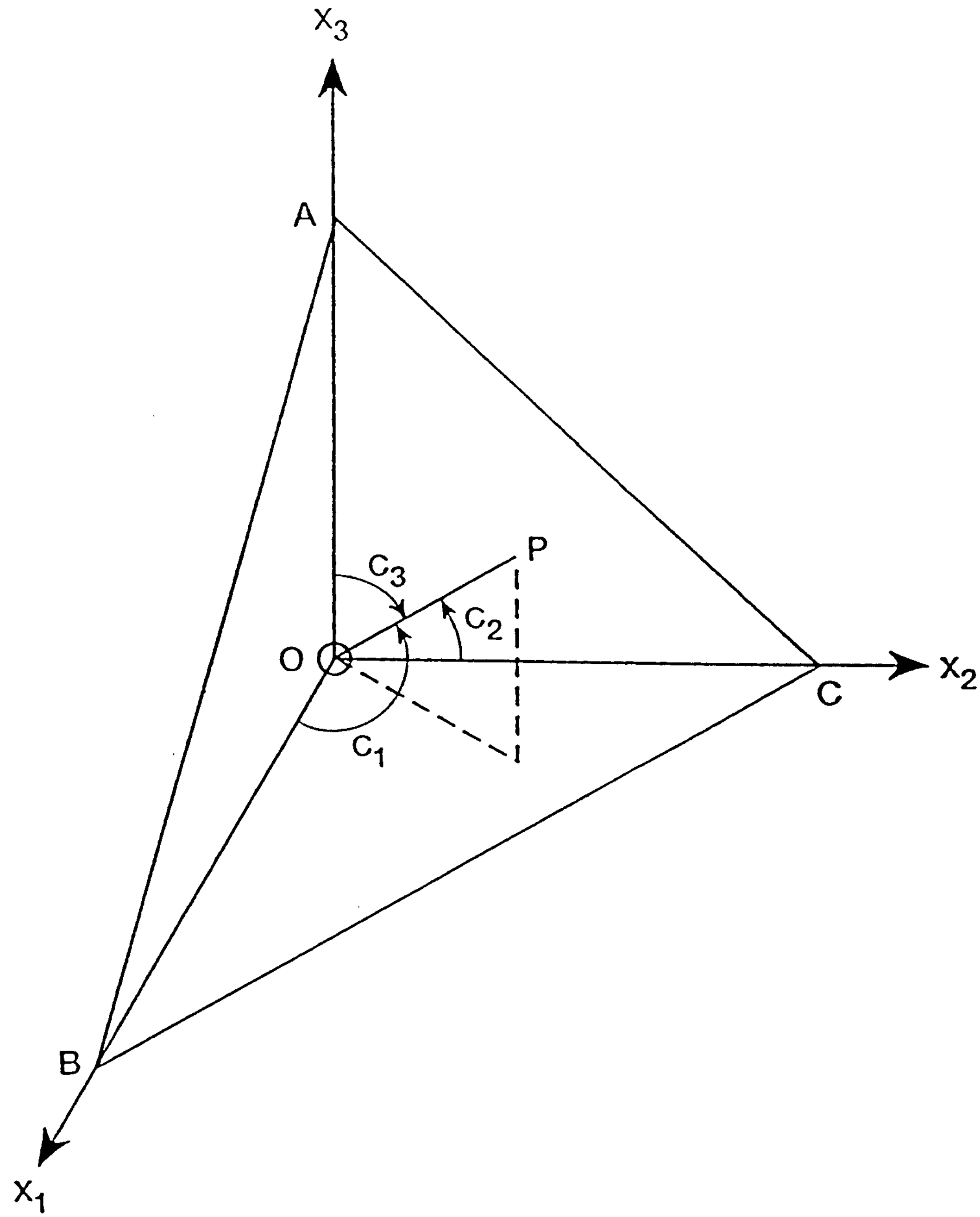


Figure 1. Geometry for a three-dimensional plane wave. Ray along line from O to P is defined by the direction cosines of the angles c_1, c_2, and c_3. The wave has the same value everywhere on the plane perpendicular to OP and passing through A, B, and C.

and origin phases, is added together, an elaborate pattern results. If the direction cosines and origin phases are randomly chosen, the pattern will be random, but will have an underlying correlation structure depending on the type of wave and method of selecting direction cosines and origin phases. Also, as the number of waves goes to infinity, the resultant field may have fractal properties.

3. CORRELATION STRUCTURE FORMULAS

To simplify the formulas, waves are first standardized so that over a wavelength the mean is zero and standard deviation is one. When m (m ≥ 2) random-independent wave fields are added together with weights $(W_I)^{1/2}$, the resultant field, F,

$$F(\underline{x}) = \sum_{i=1}^{m} W_i^{1/2}\, \mathrm{Wave}_i\, (\underline{c}_i \cdot \underline{x}) \tag{4}$$

will have a variance of one if,

$$\sum_{i=1}^{m} W_i = 1 \tag{5}$$

and the waves have been standardized. Variance can be defined in two (at least) ways for the field F. One way is to use all values of $\underline{x}$ in a single realization to calculate the variance. The other way is to calculate variance at a single point using multiple realizations. Equation (5) is true for both cases.

Given multiple realizations, suppose the correlation between the values of wave i of m waves at $\underline{x}_1$ and $\underline{x}_2$ is $r_i(\underline{x}_1, \underline{x}_2)$, and that the i waves are independent of each

other, then the correlation, r_F, between values of F at $\underline{x}_1$ and $\underline{x}_2$ is:

$$r_F = \sum_{i=1}^{m} W_i \quad r_i \tag{6}$$

Proof: The correlation between a set of standardized variables $v(\underline{x}_1)$ and $v(\underline{x}_2)$ is:

$$r = \sum_{k=1}^{K} v(\underline{x}_1) \; v(\underline{x}_2) \tag{7}$$

where the summation is over multiple, 1 to K, realizations. Combining (4) and (7):

$$r_F(\underline{x}_1, \underline{x}_2) = \sum_{k=1}^{K} \left[\sum_{i=1}^{m} W_i^{1/2} v_i(\underline{x}_1)\right]\left[\sum_{j=1}^{m} W_j^{1/2} v_j(\underline{x}_2)\right] \tag{8}$$

All cross terms ($v_I v_j$, $I \neq j$) will go to zero as K goes to infinity since the waves are independent of each other. Thus, with only J=i terms (8) becomes:

$$r_F(\underline{x}_1, \underline{x}_2) = \sum_{k=1}^{K} \sum_{i=1}^{m} [W_i^{1/2} v_i(\underline{x}_1)][W_j^{1/2} v_i(\underline{x}_2)] \tag{9}$$

$$= \sum_{i=1}^{m} W_i \sum_{k=1}^{K} v_i(\underline{x}_1) \; v_i(\underline{x}_2) \tag{10}$$

and using (7) becomes (6).

The importance of this result is that many waveforms, each with its own correlation structure, can

The importance of this result is that many waveforms, each with its own correlation structure, can be superimposed to provide a great variety of correlation structures. Further, since the correlation structure is generated from a set of realizations (ignoring the trivial case v-0 everywhere), the correlation structure resulting from a superimposition of waves will have a positive definite matrix of correlations for any set of points. Thus, any of the resulting correlation structures are eligible for consideration in objective analysis.

4. HOMOGENEOUS ISOTROPIC CORRELATION

If the correlation depends only on the distance between two points, not on where or in what direction distance is measured, the correlation structure is said to be homogeneous and isotropic. Such a structure results when a field is the result of many waves superimposed with their directions uniformly random, that is, any direction is as equally likely as any other.

One way to pick a random direction is to randomly pick a point in a sphere and to use the ray from the center of the sphere to the point to specify a direction. A random point in a sphere can be chosen by picking a random point in a cube enclosing the sphere using uniform random variables, but only using the point if it lies within the sphere. This algorithm is described in detail in Gringorten and Boehm (1987). This and other algorithms are discussed in Marsaglia (1972).

4.1. Direction Cosines

The direction cosines specify the direction of a linear wave. Given that any direction is equally probable, what is the probability density of the direction cosines?

Given the area of a sphere (in 2 dimensions "area" is the length of arc on a circle while in D dimensions "area" is the boundary of a hypersphere) less than a given direction cosine, the change in area per change in direction cosine gives the probability density, f_{cD}, that a specific direction cosine, c, would be chosen:

1	step function .5 at −1	spike at −1 and 1	(11)
2	$[\sin^{-1}(c)/\pi] + 1/2$	$1/[\pi(1 - c^2)^{1/2}$	(12)
3	$(c + 1)/2$	$1/2$	(13)
4	$\frac{\sin^{-1}(c)+c(1-c^2)^{1/2}}{\pi} + 1/2$	$\frac{2(1 - c^2)^{1/2}}{\pi}$	(14)
5	$\frac{2 + 3c - c^3}{\pi}$	$3(1 - c^2)/4$	(15)
	$I_{\frac{c+1}{2}}\left(\left(\frac{D-1}{2}\right), \left(\frac{D-1}{2}\right)\right)$	$\frac{(1 - c^2)^{[(D-3)/2]}}{B(1/2,(D-1)/2}$	(16)

where $I_x(a,b)$ is the incomplete beta function using the notation of Eq (26.5.1) of Abramowitz and Stegon (1964), and B is the beta function:

$$B[1/2,(D-1)/2] = \frac{\Gamma(D/2)}{\Gamma(1/2)\Gamma((D-1)/2)} \tag{17}$$

The left side of (11) to (16) is the integral of the right side. In the general case, (16), the form of the

incomplete beta function is obtained from the right side by noting that,

$$(1-c)^{[(d-3)/2]}(1+c)^{[(d-3)/2]} = (1-c^2)^{[(d-3)/2]} \tag{18}$$

and changing the variable of integration c=2x-1.

The direction cosine density is symmetric and therefore the odd moments are zero and the raw moments are equal to moments about the mean. Given the form of a beta distribution through the change of variable $s=1-c^2$, moments of the direction cosine density are easy because the moment integral itself is given by a beta function. The mth, moment, $\mu^{(m)}$, is:

$$\mu^{(m)} = \frac{B[(m+1)/2,(D-1)/2]}{B[1/2,(D-1)/2]} \tag{19}$$

As D becomes large, the distribution of the direction cosine approaches a normal distribution with a mean of zero and variance of 1/D.

To obtain values along a single axis in D space, it is not necessary to generate direction cosines for all the axes. Since the correlation is isotropic, any line can arbitrarily be aligned as the first axis then all terms in (2) or (3) except the phase at the origin, u_0, and the first direction cosine term are zero. In this

case, the cumulative probabilities, the left side of (11) through (16), can be used to generate a random direction cosine using a uniform random number:

To directly generate values along a second or third axis, the conditional probability must be used. The conditional density is the probability density of the next lower dimension times a factor that reduces its range,

$$f_{cD}(c_2 \mid c_1) = (1 - c_1^2)^{1/2} f_{c(D-1)} \tag{20}$$

$$f_{cD}\, c_m \mid c_{m+1}, \ldots, c_D) = (1 - \sum_{j=m+1}^{D} c_j^2)^{1/2} f_{c(D-m)} \tag{21}$$

4.2. Isotropic Correlation in D Dimensions

Consider the correlation between values at two points separated by a distance, δ, along a one dimensional wave:

$$r_1(\delta) = \{ \int_0^1 \text{Wave}(x)\ \text{Wave}(x+\delta)\ dx - \mu^2 \}/\sigma \tag{22}$$

where x and δ are in units of wavelength, μ is the mean of the wave, and δ the standard deviation. Two ways of viewing (22) are equally valid. First, consider the two points as fixed and x as the phase, i.e., integrating over all phases zero to one. The second way is to

consider the phase fixed and x as a location, that is, integrating overall locations from zero out to one wavelength.

If multiple waves are used as in (4), then the resulting correlation functions can be calculated by (6). If the probability density of a given wave is used for the weight, the resulting correlation function is an expected value, e.g., the asymptotic result of many realizations.

Further, if the probability density used is the probability density of a direction cosine as in (11) to (16) above, the resulting generality separation distance, δ, may be measured along the first axis and the first direction cosine completely defines the equivalent wavelength, 1/c, along that axis.

Correlation as a function of separation distance, δ, can be calculated by integrating [vice summing as in (6)] over all possible values of the direction cosine weighted by that value's probability density;

$$r_D(\delta) = \int_{-1}^{1} f_{cD}(c)\ r_1(\delta/c)\ dc \tag{23}$$

The integral is sometimes easier to integrate if the lower limit is set to zero and the result multiplied by 2 (because of symmetry) and the distance, s, $s=\delta/c$, is substituted as a change of variable:

$$r_D(\delta) = \frac{2}{\delta} \int_0^1 f_{cD}[s/\delta]\, r_1(s)\; ds \tag{24}$$

(24) along with (11) through (16) and (22) were used to find correlation functions for several waveforms.

4.3. Correlation of Sine Waves

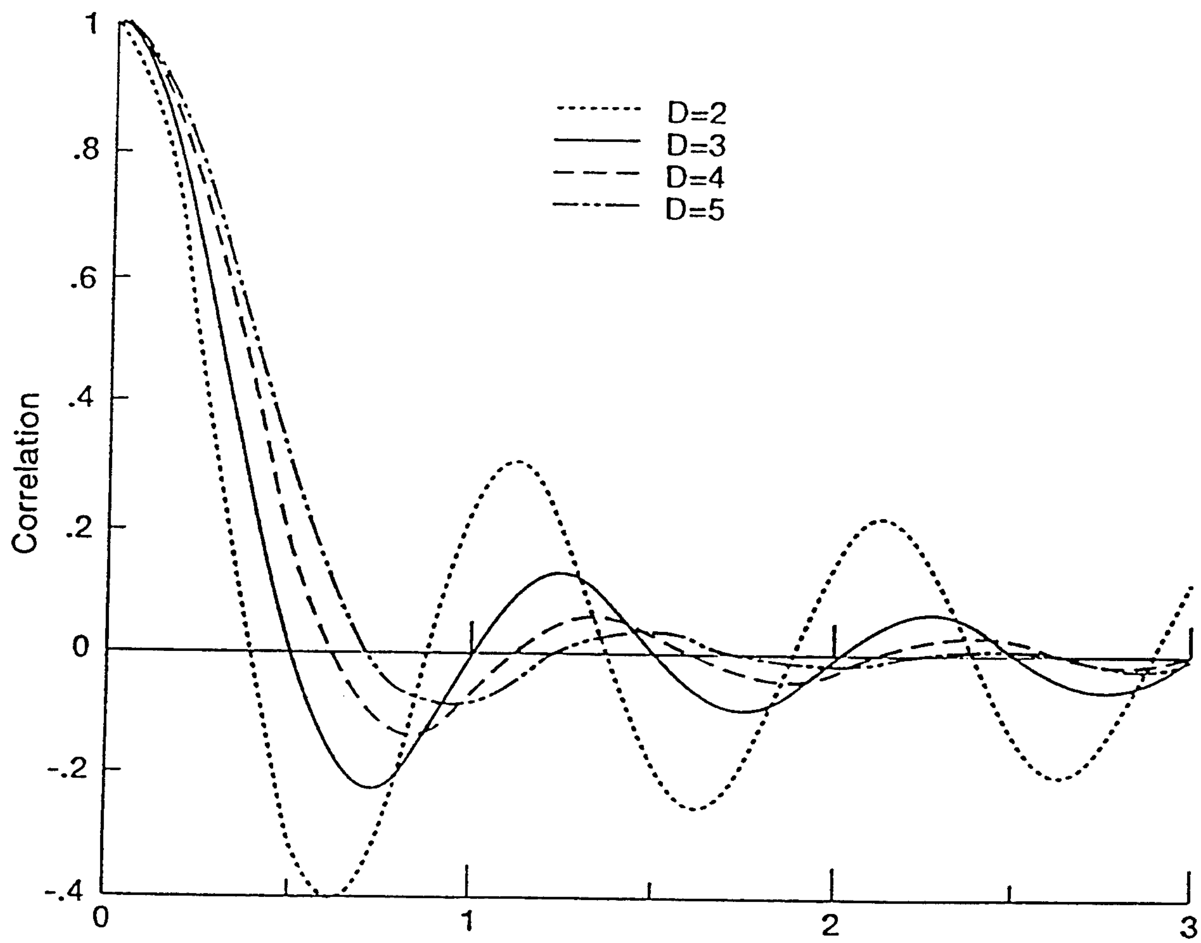
Sine and cosine waves have a mean of zero and variance of 1/2. The wavelength of sine (and cosine) waves are generally expressed in radians so that if δ is in wavelengths, arguments must be multiplied by 2π. The one-dimensional correlation is simply:

$$r_{1sin}(\delta) = \cos(2\pi\delta) \tag{25}$$

from which the higher dimensional correlations can be calculated:

$$r_{2sin}(\delta) = J_0(2\pi\delta) \tag{26}$$

$$r_{3sin}(\delta) = \frac{\sin(2\pi\delta)}{2\pi\delta} \tag{27}$$



Separation Distance σ

Sine/Cosine Wave Correlation

Figure 2. Sine or cosine wave correlation as a function of separations distance for waves with uniform distributions of direction and phase in 2, 3, 4, and 5 dimensions.

$$r_{4sin}(\delta) = 2\ J_1(2\pi\delta)/(2\pi\delta) \tag{28}$$

$$r_{5sin}(\delta) = \frac{3\sin(2\pi\delta)\ -\ 2\pi\delta\cos(2\pi\delta)}{8\pi^3\delta^3} \tag{29}$$

where J_n is the nth order Bessel function of the first kind (Abramowitz and Stegon, 1964). Bessel functions can not be evaluated directly, however, series expansions, approximations, and computer algorithms are available.

Since other waveforms can usually be represented as a series of sine and cosine waves via the Fourier transform, the correlation functions can also be represented as a series of sine correlation functions. However, integrating the forms directly most often results in a simpler form.

4.4. Correlation of Sawtooth Waves

The sawtooth wave is discontinuous at integer values of wavelength, δ, so that it is convenient to define f as fraction of wavelength (or phase):

$$f = \delta\ -\ \mathrm{int}(\delta) \tag{30}$$

where int(δ) is the largest integer in δ. Note: int is defined this way in Basic but only for positive values in

FORTRAN. The value of the sawtooth wave is equal to the phase,

$$\mathrm{Saw}(\delta) = f \tag{31}$$

Integrals are generally evaluated piecewise between integer values, so that when the integral,

$$\begin{aligned}\int_0^1 \mathrm{Saw}(x)\,\mathrm{Saw}(x+\delta)\ dx &= \int_0^{1-\delta} x(x+\delta)\ dx + \int_{1-\delta}^{1} x(x+\delta-1)\ dx \\ &= (2 - 3\delta + 3\delta^2)/6\end{aligned} \tag{32}$$

is combined with the mean of 1/2 and variance of 1/12 for the sawtooth wave the correlation is given as a function of separation distance:

$$r_{1saw}(\delta) = 1 - 6\delta + 6\delta^2, \qquad \delta \le 1 \tag{33}$$

or,

$$r_{1saw}(\delta) = 1 - 6f + 6f^2, \qquad f = \delta - \mathrm{int}(\delta) \tag{34}$$

Integration of (24) using (33) is straightforward and gives in general:

$$r_{Dsaw}(\delta \mid \delta \le 1) = 1 - \frac{12}{(D-1)B(1/2, D/2-1/2)}\delta + \frac{6}{D}\ \delta^2 \tag{35}$$

In fitting models to observed correlations, the inverse equation, i.e., distance given correlation, is sometimes needed. This distance is unambiguous only as

far as the first minimum. The lower dimensional equations, their inverses and first minimums (r|1st min) are:

$$r_{2saw}(\delta\,|\,\delta \le 1) = 1 - \frac{12}{\pi}\delta + 3\delta^2 \tag{36}$$

$$\delta_{2saw}(r\,|\,\delta < 2/\pi) = 2/\pi - [(r-1)/3 + 4/\pi^2]^{1/2} \tag{37}$$

$$r_{3saw}(\delta\,|\,\delta \le 1) = 1 - 3\delta + 2\delta^2 \tag{38}$$

$$\delta_{3saw}(r\,|\,\delta \le 3/4) = 3/4 - (r/2 + 1/16)^{1/2} \tag{39}$$

$$r_{4saw}(\delta\,|\,\delta \le 1) = 1 - \frac{8}{\pi}\delta + \frac{3}{2}\delta^2 \tag{40}$$

$$\delta_{4saw}\left(r\,|\,\delta \le \frac{8}{3\pi}\right) = \frac{8}{3\pi} - \left[2(r-1)/3 + \frac{64}{9\pi^2}\right]^{1/2} \tag{41}$$

For $\delta > 1$, values of correlation can be obtained by integrating (24) by separating δ into two parts,

$$\delta = I + f, \text{ where } I = \mathrm{int}(\delta), \text{ and } f = \delta - \mathrm{int}(\delta), \tag{42}$$

and integrating each term separately. After combining terms, the correlation is given by:

$$r_{2saw} = 1 - \frac{12}{\pi}\delta + 3\delta^2 + 6(I + I^2) - \frac{24}{\pi}\sum_{i=1}^{I}\left[i \sin^{-1}\left(\frac{i}{\delta}\right) + (\delta^2 - i^2)^{1/2}\right] \tag{43}$$

$$r_{3saw} = (f - 3f^2 + 2f^3)/\delta \tag{44}$$

$$r_{4saw} = 1 - \frac{8}{\pi}\delta + \frac{3}{2}\delta^2 + 6(I + I^2) - \frac{8}{\pi}\sum_{i=1}^{I}\left[\left(2 + \frac{i^2}{\delta^2}\right)(\delta^2 - i^2) + 3i\ \sin^{-1}\left(\frac{i}{\delta}\right)\right] \tag{45}$$

$$r_{5saw} = 1 - 2.25\delta + 1.2\,\delta^2 + 6(I^2 + I) - 4.5I\delta - 3I(I^2 + 1.5I + 1/2)/\delta + I(0.3I^4 + 0.75I^3 + 0.5I^2 - 0.05)/\delta^3 \tag{46}$$

4.5. Correlation of Triangular Waves

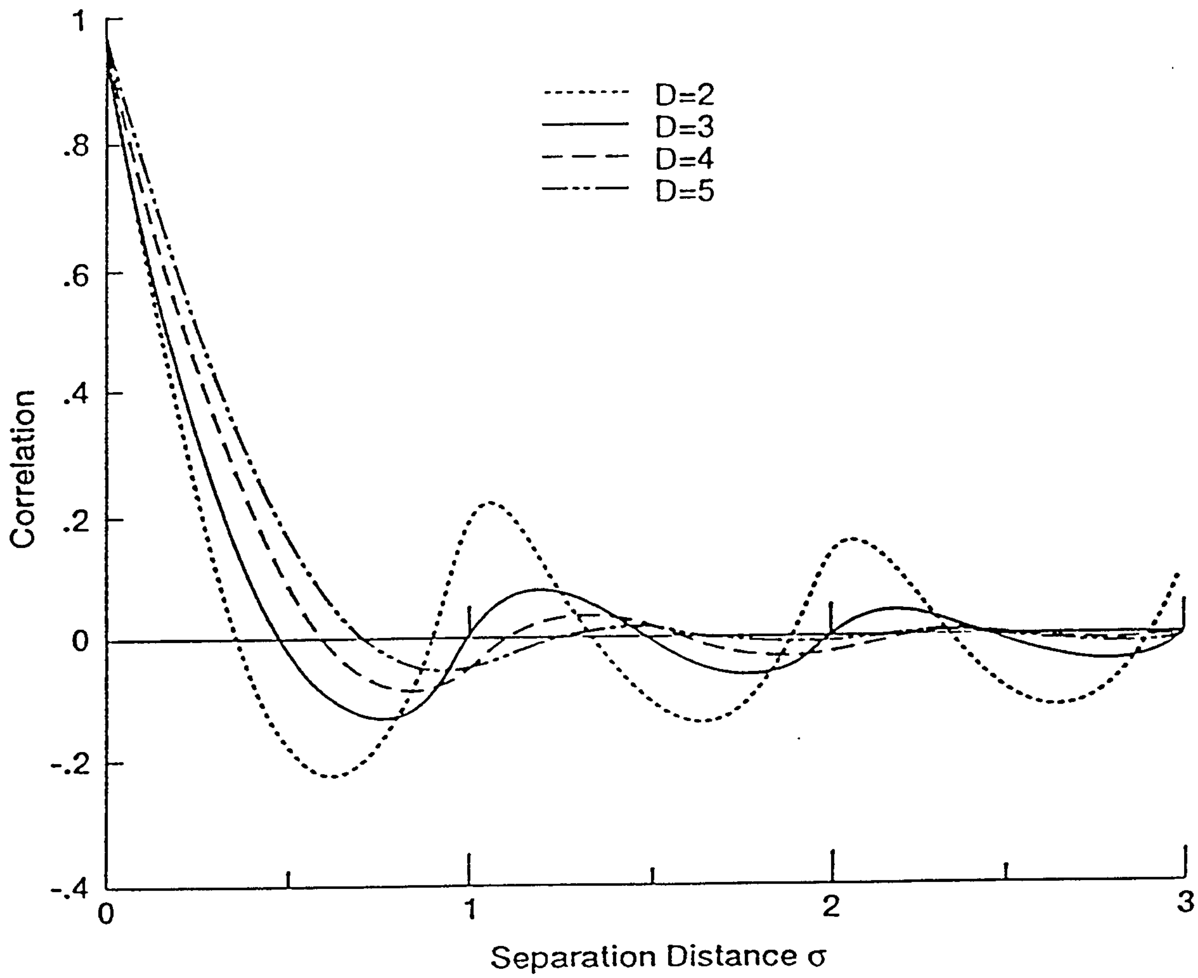
The value of the triangular wave can be written:

$$\begin{aligned} \mathrm{Tri}(\delta) &= 1 - 2f, \qquad 0 \leq f < 1/2 \\ &= 2f - 1, \qquad 1/2 \leq f < 1 \end{aligned} \tag{47}$$

with f = δ - int(δ) as above. The convolution integral can then be written:

$$\begin{aligned} \int_0^1 \mathrm{Tri}(x)\mathrm{Tri}(x+f)\ dx &= \int_0^{1/2-f} (1-2x)[1-2(x+f)]\ dx \\ &+ \int_{1/2-f}^{1/2} (1-2x)[2(x+f)-1]\ dx + \int_{1/2}^{1-f} (2x-1)[2(x+f)-1]\ dx \\ &+ \int_{1-f}^{1} (2x-1)[1-2(x-1-f)]\,dx, \qquad \text{for } f \leq 1/2 \end{aligned} \tag{48}$$

$$= 1/3 - 2\delta^2 + \frac{8}{3}\delta^3, \qquad \text{for } f \leq 1/2 \tag{49}$$



Sawtooth Wave Correlation

Figure 3. Sawtooth wave correlation as a function of separations distance for waves with uniform distributions of direction and phase in 2, 3, 4, and 5 dimensions.

A similar set of integrals for $1/2 < f \leq$ results in:

$$\int_0^1 \mathrm{Tri}(x)\mathrm{Tri}(x+\delta)\ dx = 1 - 4f + 6f^2 - \frac{8}{3}f^3, \qquad (50)$$

$$\text{for } 1/2 < f < 1$$

When these results are combined with the triangular wave's mean of 1/2 and variance of 1/12, the correlation becomes:

$$r_{1tri}(\delta \mid f \leq 1/2) = 1 - 24f^2 + 32f^3 \qquad (51)$$

$$r_{1tri}(\delta \mid 1/2 < f < 1) = 9 - 48f + 72f^2 - 32f^3 \qquad (52)$$

When (50) is put into the D-dimensional correlation formula (24), the following are obtained:

$$r_{2tri}(\delta \mid \delta \leq 1/2) = 1 - 12\delta^2 + \frac{128}{3\pi}\delta^3 \qquad (53)$$

$$r_{3tri}(\delta \mid \delta \leq 1/2) = 1 - 8\delta^2 + 8\delta^3 \qquad (54)$$

$$r_{4tri}(\delta \mid \delta \leq 1/2) = 1 - 6\delta^2 + \frac{256}{15\pi}\delta^3 \qquad (55)$$

For separation distances, δ, greater than one-half, the integral is possible but becomes increasingly complicated:

$$
\begin{aligned}
r_{2tri}(\delta) = \sum_{i=0}^{I-1} \{ & \sin^{-1}(\frac{i}{\delta})[(288i+72)\delta^2+192i^2+144i^2-6) \\
& -\sin^{-1}(\frac{i+1/2}{\delta})[(576i+288)\delta^2+384i^3+576i^2+288i+48] \\
& +\sin^{-1}(\frac{i+1}{\delta})[(288i+216)\delta^2+192i^3+432i^2+288i+54] \\
& +(\delta^2-i^2)^{1/2}(128\delta^2+352i^2+216i) \\
& +[\delta^2-(i+1/2)^2]^{1/2}[-128\delta^2-352(i^2+i)-88] \\
& +[\delta^2-(i+1)^2]^{1/2}(128\delta^2+352i^2+488i+136)\} \quad +R(f)
\end{aligned}
\tag{56}
$$

where R(f) is [Recall from (8) f = δ - int(δ) and I = int(δ)]:

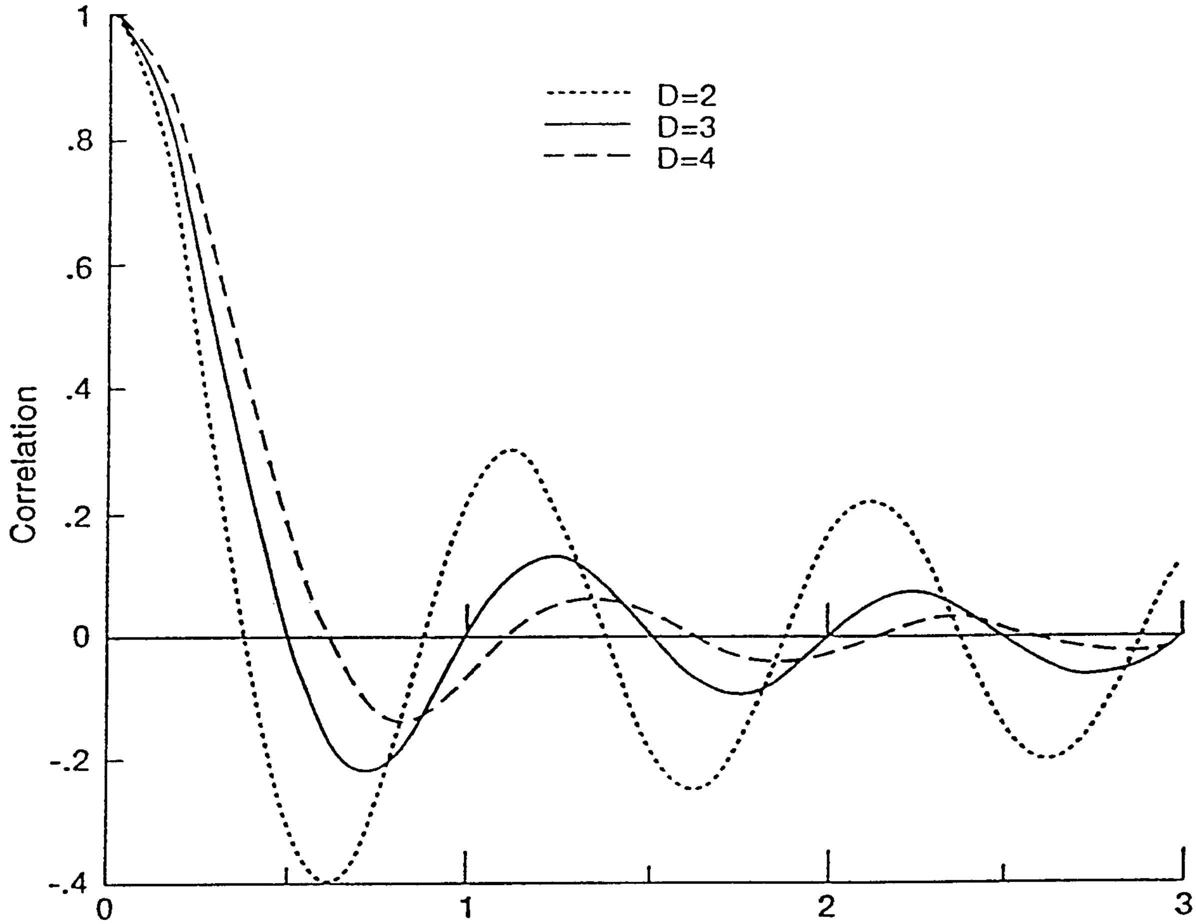
$$
\begin{aligned}
R(f\,|\,f\geq 1/2) = \; & \sin^{-1}(\frac{I}{\delta})[(288I+72)\delta^2+192I^3+144I^2-6] \\
& -\sin^{-1}(\frac{I+1/2}{\delta})[(576I+288)\delta^2+384I^3+576I^2+288I+48] \\
& +(\delta^2-I^2)^{1/2}(128\delta^2+352I^2+216I) + 2[\delta^2-(I+\frac{1}{2})^2][-128\delta^2 \\
& -352(I^2+I)-88]+\pi[(144I+108)\delta^2+96I^3|+216I^2+144I+27]
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
R(f\,|\,f<1/2) = \; & \sin^{-1}(\frac{I}{\delta})[(288I+72)\delta^2+192I^3+144I^2-6] \\
& +(\delta^2-I)^{1/2}(128\delta^2+352I^2+216I)-\pi[(144I+36)\delta^2+96I^3+72I^2-3]
\end{aligned}
\tag{58}
$$

The three-dimensional triangular wave correlation function is easier because of the simpler form of the three-dimensional directional cosine density:

$$r_{3tri}(\delta \,|\, f \leq 1/2) = (f - 8f^3 + 8f^4)/\delta \tag{59}$$

$$r_{3tri}(\delta \,|\, f > 1/2) = -[(1-f) - 8(1-f)^3 + 8(1-f)^4]/\delta \tag{60}$$



Separation Distance σ

Triangular Wave Correlation

Figure 4. Triangular wave correlation as a function of separations distance for waves with uniform distributions of direction and phase in 2, 3, and 4 dimensions.

5. DISCUSSION

This report presents the method of calculating correlation functions for periodic functions - waves - in D-dimensional space. Explicit formulas for correlation functions of several important waveforms are given. These waveforms include the sine/cosine, the sawtooth wave, and the triangular wave.

The correlation function for triangular waves is nearly the same for as that for sine waves. When the graph for one is placed on top of graph for the other, the graphs appear to be identical. This similarity has two important applications. In simulating a field that has a sine type of correlation, the triangular wave can be used; the triangular wave being an order of magnitude faster to simulate than a sine wave.

The second application is in approximating the first order Bessel function. The 2D sawtooth correlation is particularly applicable in inverse Bessel function.

Often objective analysis requires correlation functions of a certain shape. The sawtooth has a correlation similar to $\exp(-\delta^2)$. By mixing the sawtooth

and triangular wave correlation curves and by also mixing the wavelengths, a variety of correlations is possible. These correlation curves are guaranteed to regression using correlations will always (in theory) be able to be inverted. Further, objective analysis is mainly concerned with close (less than one δ) points. For these points, the resulting mixture is a simple quadratic polynomial that is very fast to calculate.

At first, sawtooth correlation functions were only available as the result of numerous simulations. When these were to be combined to approximate another correlation function, numerous simulation runs were required. In order to see the effect of varying different weights and wavelengths, graphs from these different runs were placed side to side and top to bottom. They filled the whole floor of a room. What a help the equations in this report could have been then.

6. REFERENCE

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